

On the Laplace Normal Vector Field of Skew Ruled Surfaces

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Abstract

We consider the Laplace normal vector field of relatively normalized ruled surfaces with non-vanishing Gaussian curvature in the Euclidean space \mathbb{R}^3 . We determine all ruled surfaces and all relative normalizations for which the Laplace normal image degenerates into a point or into a curve. Moreover, we study the Laplace normal image of a non-conoidal ruled surface whose relative normals lie on the asymptotic plane.

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1 Introduction

Relatively normalized ruled surfaces with non-vanishing Gaussian curvature in the Euclidean space \mathbb{R}^3 have been studied in the last years in many points of view (see [2], [4], [5], [6]; for more details and references see [8]). This paper deals with the Laplace normal vector field of skew ruled surfaces. We first show that for any relative normalization $\bar{\gamma}$ the Laplace normal vector of a ruled surface Φ along each ruling lies on the corresponding asymptotic plane. Then, we determine all ruled surfaces and all relative normalizations, for which the Laplace normal image degenerates into a point or into a curve. We finish the paper by the study of the Laplace normal image of a non conoidal ruled surface such relatively normalized, that the relative normals $\bar{\gamma}$ along each ruling lie on the corresponding asymptotic plane.

2 Preliminaries

To set the stage for this work the classical notations of relative differential geometry and of ruled surfaces theory are briefly presented. For this purpose the books [3] and [7] are used as general references.

In the Euclidean space \mathbb{R}^3 let Φ be a skew (non-developable) ruled C^r -surface, $r \geq 2$. We denote by $\bar{s}(u)$, $u \in I$ ($I \subset \mathbb{R}$ open interval) the position vector of the line of striction of Φ and by $\bar{e}(u)$ the unit vector pointing along the rulings. We choose the parameter u to be the arc length along the spherical curve $\bar{e}(u)$. Then a parametrization of the ruled surface Φ over the region $U := I \times \mathbb{R}$ of the (u, v) -plane is

$$\bar{x}(u, v) = \bar{s}(u) + v\bar{e}(u) \quad \text{with} \quad |\bar{e}| = |\bar{e}'| = 1, \quad \langle \bar{s}'(u), \bar{e}'(u) \rangle = 0 \text{ in } I, \quad (1)$$

where the prime denotes differentiation with respect to u and $\langle \cdot, \cdot \rangle$ the standard scalar product in \mathbb{R}^3 . The *Kruppa frame* of Φ , consisting of the vector $\bar{e}(u)$, the central normal vector $\bar{n} := \bar{e}'$ and the central tangent vector $\bar{z} := \bar{e} \times \bar{n}$, satisfies the relations [3, p. 17]

$$\bar{e}' = \bar{n}, \quad \bar{n}' = -\bar{e} + \kappa\bar{z}, \quad \bar{z}' = -\kappa\bar{n}, \quad (2)$$

where

$$\kappa = (\bar{e}, \bar{e}', \bar{e}'')$$

denotes the *conical curvature* of Φ . Consider the *parameter of distribution*

$$\delta = (\bar{s}', \bar{e}, \bar{e}')$$

and the *striction*

$$\sigma := \angle(\bar{e}, \bar{s}') \quad \left(-\frac{\pi}{2} < \sigma \leq \frac{\pi}{2}, \text{sign } \sigma = \text{sign } \delta\right)$$

of Φ . Then, the tangent vector \bar{s}' of the line of striction and the unit normal vector $\bar{\xi}$ of Φ are expressed by

$$\bar{s}' = \delta(\lambda\bar{e} + \bar{z}), \quad (3)$$

$$\bar{\xi} = \frac{\delta\bar{n} - v\bar{z}}{w}, \quad (4)$$

where

$$\lambda := \cot \sigma$$

and

$$w^2 := v^2 + \delta^2.$$

When the *fundamental invariants* $\kappa(u)$, $\delta(u)$ and $\lambda(u)$ are given, then there exists up to rigid motions of the space \mathbb{R}^3 a unique ruled surface Φ , whose fundamental invariants are the given.

The components h_{ij} of the second fundamental form in the coordinates $u^1 := u$, $u^2 := v$ are the following

$$h_{11} = -\frac{\kappa v^2 + \delta'v + \delta^2(\kappa - \lambda)}{w}, \quad h_{12} = \delta, \quad h_{22} = 0. \quad (5)$$

The Gaussian curvature K of Φ is given by

$$K = \frac{-\delta^2}{w^4}. \quad (6)$$

A C^s -mapping $\bar{y} : U \longrightarrow \mathbb{R}^3$, $r > s$, is called a C^s -relative normalization of Φ if

$$\text{rank} \{ \bar{x}_{/1}, \bar{x}_{/2}, \bar{y} \} = 3, \quad \text{rank} \{ \bar{x}_{/1}, \bar{x}_{/2}, \bar{y}_{/1}, \bar{y}_{/2} \} = 2 \quad \forall \alpha \in U. \quad (7)$$

The *covector* \bar{X} of the tangent plane is defined by

$$\langle \bar{X}, \bar{x}_{/1} \rangle = \langle \bar{X}, \bar{x}_{/2} \rangle = 0 \quad \text{and} \quad \langle \bar{X}, \bar{y} \rangle = 1. \quad (8)$$

The *relative metric* G on U is introduced by

$$G_{ij} = \langle \bar{X}, \bar{x}_{/ij} \rangle. \quad (9)$$

The *support function* of the relative normalization \bar{y} is defined, according to [6], by

$$q := \langle \bar{\xi}, \bar{y} \rangle : U \longrightarrow \mathbb{R}, \quad q \in C^s(U),$$

where $\bar{\xi} : U \longrightarrow \mathbb{R}^3$ is the *Euclidean normalization* of Φ . By virtue of (7), q never vanishes on U ; moreover, because of (8), it turns out

$$\bar{X} = q^{-1} \bar{\xi}. \quad (10)$$

On account of (9) and (10) we obtain

$$G_{ij} = q^{-1} h_{ij}. \quad (11)$$

We mention that given a support function q , the relative normalization \bar{y} is uniquely determined and possesses the following parametrization [6, p. 197]

$$\bar{y} = -h^{ij} q_{/i} \bar{x}_{/j} + q \bar{\xi}, \quad (12)$$

where h^{ij} are the components of the inverse tensor of h_{ij} .

In [2], H. Heil has introduced the C^1 -mapping $\bar{L} : U \longrightarrow \mathbb{R}^3$, defined by

$$\bar{L}(u, v) = \frac{\Delta \bar{x}(u, v)}{2}, \quad (13)$$

where Δ is the laplacian with respect to the relative metric G , as *the Laplace normal vector* of Φ . When we move the Laplace normal vectors to a fixed point, the endpoints of them describe the *Laplace normal image* of Φ .

¹Partial derivatives of a function f are denoted by $f_{/i} := \frac{\partial f}{\partial u^i}$, $f_{/ij} := \frac{\partial^2 f}{\partial u^i \partial u^j}$ etc.

3 The Laplace normal vector field of a ruled surface

We consider a relatively normalized skew ruled C^2 -surface $\Phi \subset \mathbb{R}^3$. By virtue of (1)-(5) and (12), the relative normalization \bar{y} can be written as follows

$$\bar{y} = -w \frac{\delta q_{/1} + q_{/2}(\kappa w^2 + \delta' v)}{\delta^2} \bar{e} + \frac{\delta^2 q - w^2 v q_{/2}}{\delta w} \bar{n} - \frac{v q + w^2 q_{/2}}{w} \bar{z}. \quad (14)$$

We denote by

$$q_{AFF} := |K|^{1/4} \quad (15)$$

the support function of the *equiaffine normalization* \bar{y}_{AFF} of Φ . On account of (6), (14) and (15) we get

$$\bar{y}_{AFF} = \frac{\varepsilon}{|\delta|^{1/2}} \left(\frac{2\kappa v + \delta'}{2\delta} \bar{e} + \bar{n} \right), \quad (16)$$

where $\varepsilon = \text{sign } \delta$. Using (1)-(5), (11) and (13) we find

$$\bar{L} = \frac{wq}{\delta} \left(\frac{2\kappa v + \delta'}{2\delta} \bar{e} + \bar{n} \right), \quad (17)$$

i.e.

$$\bar{L} = \frac{q}{q_{AFF}} \bar{y}_{AFF}$$

which shows, that for any normalization \bar{y} the Laplace normal \bar{L} of Φ along each ruling lies on the corresponding asymptotic plane in the direction of the equiaffine normalization \bar{y}_{AFF} and therefore it is independent of the relative normalization \bar{y} .

Remark. Obviously, two ruled surfaces parametrized by (1) with parallel rulings, common parameter of distribution and common support function have the same Laplace normal vector field.

We shall firstly determine all ruled surfaces, whose Laplace normals are constant along each ruling. We put

$$L_1 = \frac{wq(2\kappa v + \delta')}{2\delta^2}, \quad L_2 = \frac{wq}{\delta}, \quad (18)$$

and on account of (2) we find

$$\bar{L}_{/1} = (L_{1/1} - L_2) \bar{e} + (L_1 + L_{2/1}) \bar{n} + \kappa L_2 \bar{z}, \quad \bar{L}_{/2} = L_{1/2} \bar{e} + L_{2/2} \bar{n}. \quad (19)$$

Thus we have $\bar{L}_{/2} = \bar{0}$ if and only if κ vanishes and the support function is of the form

$$q = \frac{f(u)}{w}, \quad (20)$$

where $f(u)$ is an arbitrary nonvanishing C^2 -function. So we have the

Proposition 1 *The Laplace normals of a relatively normalized ruled C^3 -surface Φ , which is free of torsal rulings, are constant along each ruling if and only if Φ is conoidal and the support function is of the form (20).*

In this case the Laplace normal becomes

$$\bar{L}(u) = \frac{f}{\delta} \left(\frac{\delta'}{2\delta} \bar{e} + \bar{n} \right), \quad (21)$$

i.e., the Laplace normal image of Φ is degenerated into a point or into a curve. In the following we investigate all ruled surfaces, which possess the previous property.

Case I. The Laplace normal image degenerates into a point if and only if $\text{rank}(\bar{L}_{/1}, \bar{L}_{/2}) = 0$, or, on account of (19), equivalently,

$$L_{1/1} - L_2 = L_1 + L_{2/1} = \kappa L_2 = L_{1/2} = L_{2/2} = 0. \quad (22)$$

Thus $\kappa = 0$ and the support function is of the form (20). In view of (18) the remaining relations of (22) become

$$\left(\frac{\delta' f}{\delta^2} \right)' - 2 \frac{f}{\delta} = 0, \quad \frac{\delta' f}{\delta^2} + 2 \left(\frac{f}{\delta} \right)' = 0. \quad (23)$$

whence

$$\left(\frac{f}{\delta} \right)'' + \frac{f}{\delta} = 0.$$

Consequently

$$\frac{f}{\delta} = c_1 \cos u + c_2 \sin u,$$

where $c_1, c_2 = \text{const.}$ Then, the second relation of (23) implies

$$\frac{\delta'}{2\delta} = \frac{c_1 \sin u - c_2 \cos u}{c_1 \cos u + c_2 \sin u},$$

from which we find

$$\delta = c_3 (c_1 \cos u + c_2 \sin u)^{-2},$$

where $c_3 = \text{const.} \neq 0$. Thus

$$f = \pm (c_3 \delta)^{1/2}.$$

On account of (6), (15) and (20) we have

$$q = \pm |c_3|^{1/2} q_{AFF}.$$

Finally from (14) and (21) we find

$$\bar{y} = \bar{L} = \bar{a},$$

where \bar{a} is the constant vector

$$\bar{a} = (c_1 \sin u - c_2 \cos u) \bar{e} + (c_1 \cos u + c_2 \sin u) \bar{n}.$$

Therefore Φ is an improper relative sphere, see [1], [7]. So we have

Proposition 2 *The Laplace normal image of a relatively normalized ruled C^3 -surface Φ , which is free of torsal rulings, degenerates into a point if and only if Φ is an improper conoidal relative sphere, the parameter of distribution is given by $\delta = c_3 (c_1 \cos u + c_2 \sin u)^{-2}$, where $c_1, c_2, c_3 = \text{const.}$, $c_3 \neq 0$, and the support function is given by $q = \pm |c_3|^{1/2} q_{AFF}$.*

Case II. The Laplace normal image degenerates into a curve Γ if and only if $\text{rank}(\bar{L}_{/1}, \bar{L}_{/2}) = 1$, or, on account of (19) equivalently,

$$\frac{L_{1/1} - L_2}{L_{1/2}} = \frac{L_1 + L_{2/1}}{L_{2/2}} = \frac{\kappa L_2}{0}. \quad (24)$$

Obviously $\kappa = 0$.

Subcase IIa. Let $L_{2/2} = 0$. The support function has therefore the form (20) and the curve Γ has the parametrization (21). Moreover it is easy to confirm, that the curve Γ is planar.

Subcase IIa. For $L_{2/2} \neq 0$ it turns out from (24)

$$2\delta\delta'' - 3\delta'^2 - 4\delta^2 = 0, \quad (25)$$

from which we obtain

$$\delta = c_2 \cos^{-2}(u + c_1), \quad c_1, c_2 = \text{const.} \quad (26)$$

Hence, a parametrization of the curve Γ is

$$\bar{L} = \frac{wq}{c_2} \cos(u + c_1) \bar{\alpha}, \quad (27)$$

where $\bar{\alpha}$ is the constant unit vector

$$\bar{\alpha} = \sin(u + c_1) \bar{e} + \cos(u + c_1) \bar{n}.$$

Obviously, Γ is a straight line. Finally, from (16) it follows

$$\bar{y}_{AFF} = \pm |c_2|^{-1/2} \bar{a},$$

so that the affine normal image of Φ is an improper affine sphere.

Thus we have proved

Proposition 3 *The Laplace normal image of a relatively normalized ruled C^3 -surface Φ , which is free of torsal rulings, degenerates into a curve Γ if and only if Φ is conoidal and either the support function has the form $q = f(u)w^{-1}$, or the parameter of distribution is given by $\delta = c_2 \cos^{-2}(u + c_1)$, where $c_1, c_2 = \text{const.}$ The curve Γ is planar. In particular in the second case, the affine normal image of Φ is an improper sphere and the curve Γ is a straight line.*

We consider a conoidal surface Φ and a support function q of the form (20). For the curvature of the planar curve Γ we find

$$k = \frac{Aff'' - 2\delta^2 Af'^2 + Bff' + Cf^2}{D}, \quad (28)$$

where

$$\begin{aligned} A &= \delta^2 (2\delta\delta'' - 3\delta'^2 - 4\delta^2), \\ B &= 2\delta (6\delta\delta'\delta'' - 4\delta^2\delta' - 6\delta'^3 - \delta^2\delta'''), \\ C &= 4\delta^4 + 7\delta^2\delta'^2 + 6\delta'^4 - 2\delta^3\delta'' - 6\delta\delta'^2\delta'' + \delta^2\delta'\delta'', \\ D &= \left[\delta^2 (\delta'f - 2\delta f')^2 + [2(\delta^2 + \delta'^2)f - \delta(\delta'f' + \delta''f)]^2 \right]^{3/2}. \end{aligned}$$

The curve Γ is a straight line if and only if the function $f(u)$ fulfills the differential equation

$$Aff'' - 2\delta^2 Af'^2 + Bff' + Cf^2 = 0.$$

The existence of ruled surfaces, whose Laplace normal image indeed degenerates into a straight line, whenever the support function is of the form (20), is guaranteed by the following examples.

1. Let Φ be a conoidal surface with

$$\delta = \text{const.} \neq 0.$$

From (28) we obtain for the curvature of Γ

$$k = \frac{|\delta(-ff'' + 2f'^2 + f^2)|}{(f'^2 + f^2)^{3/2}}.$$

Therefore Γ is a straight line if and only if

$$ff'' - 2f'^2 - f^2 = 0,$$

whose general solution, after a reparametrization, is

$$f(u) = \frac{c}{\cos u}, \quad c = \text{const.} \neq 0.$$

Thus, *the Laplace normal image of a conoidal surface with constant parameter of distribution degenerates into a straight line if and only if the support function is given by*

$$q(u, v) = \frac{c}{\cos u w}, \quad c = \text{const.} \neq 0.$$

2. Let Φ be a conoidal surface with

$$\delta = \sin^2 u.$$

From (28) we obtain for the curvature of Γ

$$k = 4\sqrt{2} \sin^4 u \frac{|(6 - 5 \sin^2 u) f^2 + 2 \sin^2 u f'^2 - f(3 \sin 2u f' + \sin^2 u f'')|}{|(5 \cos 2u + 13) f^2 - 6 \sin 2u f f' + 2 \sin^2 u f'^2|^{3/2}}.$$

The curve Γ is a straight line if and only if

$$f(3 \sin 2u f' + \sin^2 u f'') + (5 \sin^2 u - 6) f^2 - 2 \sin^2 u f'^2 = 0,$$

whose general solution is

$$f(u) = \frac{c \sin^3 u}{\cos 2u}, \quad c = \text{const.}, \neq 0.$$

Therefore, the Laplace normal image of a conoidal surface with parameter of distribution given by $\delta = \sin^2 u$ degenerates into a straight line if and only if the support function is given by

$$q(u, v) = \frac{c \sin^3 u}{\cos 2u w}, \quad c = \text{const.} \neq 0.$$

4 Ruled surfaces, whose Laplace normal image is a ruled surface

In this paragraph we deal with non conoidal ruled surfaces, whose relative normals \bar{y} along each ruling lie on the corresponding asymptotic plane. On account of (14) it is easy to see, that this occur if and only if

$$vq + w^2 q_{/2} = 0,$$

which gives that the support function is of the form (20), i.e.

$$q = \frac{f(u)}{w},$$

where $f(u)$ is an arbitrary nonvanishing C^2 -function.

Thus,

$$\Phi^* : \bar{L} = \frac{f}{\delta} \bar{n} + \frac{f(2\kappa v + \delta')}{2\delta^2} \bar{e}$$

is a parametrization of the Laplace normal image Φ^* of Φ . We see immediately, that Φ^* is a ruled surface too, whose rulings are parallel to those of Φ . So, Φ and Φ^* possess common conical curvature and common Kruppa moving frame $\{\bar{e}, \bar{n}, \bar{z}\}$.

We consider the directrix

$$\Gamma^* : \quad \bar{r}^* = \frac{f}{\delta} \bar{n} \tag{29}$$

of Φ^* . Then, it is easy to prove the following

Proposition 4 *The tangents of the line of striction Σ of Φ and the tangents of the curve Γ^* of Φ^* at corresponding points are*

a) parallel if and only if

$$f = c\delta, \text{ where } c = \text{const.}, \quad (30)$$

and

$$\kappa\lambda + 1 = 0, \quad (31)$$

i.e., the support function is given by

$$q = \frac{c\delta}{w}, \quad (32)$$

and the line of striction Σ of Φ is a line of curvature.

b) orthogonal if and only if

$$\kappa = \lambda, \quad (33)$$

i.e., the line of striction Σ of Φ is an asymptotic line.

A parametrization of the line of striction Σ^* of Φ^* can be found on account of the relation $\bar{s}^* = \bar{r}^* - \langle \bar{r}^{*\prime}, \bar{e}' \rangle \bar{e}$. We obtain

$$\bar{s}^* = \frac{f}{\delta} \bar{n} - \left(\frac{f}{\delta} \right)' \bar{e}. \quad (34)$$

From (29) and (34) we have:

Proposition 5 *The curve Γ^* coincides with the line of striction Σ^* of Φ^* if and only if the support function is given by*

$$q = \frac{c\delta}{w}, \text{ where } c = \text{const.}$$

From (34) we find

$$\bar{s}^{*\prime} = - \left[\left(\frac{f}{\delta} \right)'' + \frac{f}{\delta} \right] \bar{e} + \frac{\kappa f}{\delta} \bar{z}.$$

Consequently (see (3)), the fundamental invariants of Φ^* are the following

$$\delta^* = \frac{\kappa f}{\delta}, \quad \kappa^* = \kappa, \quad \lambda^* = \frac{-\delta}{\kappa f} \left[\left(\frac{f}{\delta} \right)'' + \frac{f}{\delta} \right]. \quad (35)$$

From (35) we obtain a series of results, which are contained in the following

Proposition 6 *(a) The Euclidean normals of Φ and Φ^* are parallel if and only if*

$$q = c q_{AFF}, \text{ where } c = \text{const.}$$

(b) The Laplace normal image Φ^* of Φ is orthoid ($\lambda^* = 0$) if and only if the support function is given by

$$q = \frac{\delta (c_1 \cos u + c_2 \sin u)}{w}, \text{ where } c_1, c_2 = \text{const.}$$

(c) The line of striction Σ^* of Φ^* is an asymptotic line ($\kappa^* = \lambda^*$) if and only if

$$\delta \left(\frac{f}{\delta} \right)'' + f (\kappa^2 + 1) = 0. \quad (36)$$

(d) The line of striction Σ^* of Φ^* is a line of curvature ($1 + \kappa^* \lambda^* = 0$) if and only if the support function is given by

$$q = \frac{\delta (c_1 + c_2 u)}{w}, \text{ where } c_1, c_2 = \text{const.}$$

(e) Φ and Φ^* are congruent if and only if the support function is given by

$$q = \frac{\delta^2}{\kappa w}$$

and the fundamental invariants of Φ are associated with the relation

$$\lambda = -\frac{1}{\delta} \left(\frac{\delta}{\kappa} \right)'' - \frac{1}{\kappa}.$$

(f) The Laplace normal image Φ^* is an Edlinger surface ($\delta^* = \text{const.}, \kappa^* \lambda^* + 1 = 0$) if and only if the support function and the conical curvature are given by

$$q = \frac{\delta (c_1 + c_2 u)}{w}, \quad \kappa = \frac{c_3}{c_1 + c_2 u}, \text{ where } c_1, c_2, c_3 = \text{const.}$$

respectively.

The following example guarantees, that there exist ruled surfaces, which realize case (c) of the above proposition, i.e. ruled surfaces, such that the line of striction Σ^* of their Laplace normal image Φ^* is an asymptotic line: For

$$\kappa = c_1 = \text{const.}$$

we obtain from (36)

$$f = \delta (c_3 \cos c_2 u + c_4 \sin c_2 u), \quad c_3, c_4 = \text{const.},$$

where $c_2 = \sqrt{c_1^2 + 1}$. Consequently, when the support function is of the form

$$q = \frac{\delta (c_3 \cos c_2 u + c_4 \sin c_2 u)}{w}, \quad c_3, c_4 = \text{const.}, \quad c_2 = \sqrt{c_1^2 + 1},$$

then, the line of striction Σ^* of Φ^* is an asymptotic line.

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